Announcements

1) Math Advising Session
11:30 - 1:30 CB 2047
(Math Library) - There
will be pizza!

2) Math Awards Ceremony
2:00, CB 2046

3) Survey: 8 complete

Recall: (orthogonal projection)

P in $M_n(\mathbb{R})$ is an orthogonal projection if $P^2 = P$ and $P = P^t$.

Example 1: $(E_{i,i})$

$E_{i,j}$ = matrix with a one in (i,j) entry and zeros in all other entries.

$E_{i,i}$ is an orthogonal projection for all $1 \le i \le n$.

$n=2$

$$E_{1,1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$E_{2,2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Note also that

$$E_{1,1} + E_{2,2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is also an orthogonal projection.

Fact: Any orthogonal projection is equal to SDS^t where S is orthogonal and D is a sum over some of the $E_{i,i}$'s.

Example 2: (non orthogonal projection)

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = A.$$

$$A^t = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \neq A$$

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x+y \\ 0 \end{bmatrix}$$

$\mathrm{Ran}(A) = x$-axis in $\mathbb{R}^2$.

But $A\begin{bmatrix} x \\ y \end{bmatrix}$ is <u>not</u>

at a right angle to

$$\begin{bmatrix} x \\ y \end{bmatrix} - A\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x+y \\ 0 \end{bmatrix} = \begin{bmatrix} -y \\ y \end{bmatrix}$$

Theorem: (form of orthogonal projection)

Let P be an orthogonal projection in $M_n(\mathbb{R})$, let $V = \operatorname{ran}(P)$. If x is in $\mathbb{R}^n$ and $\{v_1, \ldots, v_k\}$ is an orthogonal basis for V, then

$$\boxed{Px = \frac{x \cdot v_1}{\|v_1\|_2^2} v_1 + \frac{x \cdot v_2}{\|v_2\|_2^2} v_2 + \cdots + \frac{x \cdot v_k}{\|v_k\|_2^2} v_k}$$

Fact: If $V \subseteq \mathbb{R}^n$ is a subspace, then there is an orthogonal projection onto V.

Observation: $(I_n - P)$

Let P be an orthogonal projection in $M_n(\mathbb{R})$.

Then so is $I_n - P$:

$$(I_n - P)^t = I_n^t - P^t$$
$$= I_n - P \checkmark \quad \text{and}$$

$$(I_n - P)^2 = (I_n - P)(I_n - P)$$
$$= I_n - P - P + P^2$$
$$= I_n - P - P + P = I_n - P \checkmark$$

Notation: $(\perp)$

If P is an orthogonal projection, we let $P^{\perp}$ denote $I - P$. Similarly, if $V \leq \mathbb{R}^n$ is a subspace,

$V^{\perp}$ = the subspace of $\mathbb{R}^n$ consisting of all vectors that are orthogonal to every vector in V = $P^{\perp}(\mathbb{R}^n)$

Decomposition:

Let x be in $\mathbb{R}^n$, and let $V \subseteq \mathbb{R}^n$ be a subspace.

Let P = orthogonal projection onto V. Then we can write

$$x = \underbrace{Px}_{y} + \underbrace{P^\perp x}_{z}$$

Moreover, if $x = y_1 + z_1$ with y_1 in V, z_1 in $V^\perp$, then $y = y_1$, $z = z_1$

Uniqueness: Suppose

$x = y_1 + z_1$ with y_1 in

V and z_1 in $V^\perp$. Then

$$
\begin{aligned}
Px &= P(y_1 + z_1) \\
&= Py_1 + Pz_1 \quad \text{(linearity)} \\
&= P(Py_1) + \underbrace{P(P^\perp z_1)}_{=0} \\
&= P^2 y_1 = Py_1 = y_1 .
\end{aligned}
$$

Similarly, $P^\perp x = z_1$.

Observation: (2-norm)

If x is in $\mathbb{R}^n$ and P is an orthogonal projection, then Px is the closest vector in $\mathrm{Ran}(P)$ to x in $\|\cdot\|_2$:

$$\boxed{\|Px - x\|_2 \leq \|w - x\|_2}$$

for all w in $\mathrm{Ran}(P)$.

$w = Px$ minimizes the distance from $\mathrm{Ran}(P)$ to x.

If x is not in $\mathrm{Ran}(P)$, then you will never be able to solve

$Ps = x$ for som

s in $\mathbb{R}^n$.

Least Squares

(Section 6.5)

Solving $Ax=b$

where A is $m \times n$.

Given a b, you might not always be able to find an x with $Ax=b$!

Here, we assume A is not invertible.

Example 3:

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix} \quad b = \begin{bmatrix} -5 \\ 8 \\ 2 \end{bmatrix}$$

Is there a vector $\begin{bmatrix} x \\ y \end{bmatrix}$

with

$$A \begin{bmatrix} x \\ y \end{bmatrix} = b \,?$$

$$\begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 8 \\ 2 \end{bmatrix}$$

$$\mathrm{Rref}(A\ b)$$

$$= \mathrm{rref}\begin{bmatrix} 2 & 1 & -5 \\ -2 & 0 & 8 \\ 2 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Means no solution!

So we can't solve $Ax = b$

for this A and b.

We'll shift to finding vector(s) v that minimize

$$\|Av - b\|_2 .$$

Definition: (least squares approximation)

Let A be an $m \times n$ matrix and let b be in $\mathbb{R}^m$.

Then the set of **least squares** solutions to $Av = b$ is all vectors in $\mathbb{R}^n$ that minimize $\|Av - b\|_2$; i.e.

$$\boxed{\|Av - b\|_2 \leq \|Aw - b\|_2}$$

for all w in $\mathbb{R}^n$

Theorem: (how to find)

The set of least squares solutions to $Av=b$ is precisely the set of all solutions to $\boxed{A^tAv = A^tb}$

Therefore, if $A^t A$ is invertible, there is a unique v with

$$A^t A v = A^t b,$$

$$\boxed{v = (A^t A)^{-1} A^t b}$$

Back to Example 3:

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix} \quad b = \begin{bmatrix} -5 \\ 8 \\ 2 \end{bmatrix}$$

$$A^tA = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 8 \\ 8 & 10 \end{bmatrix}$$

$\det(A^tA) = 120 - 64 \neq 0$,

so A^tA is invertible.

Therefore, the unique least squares solution to $Av = b$ is

$$v = (A^t A)^{-1} A^t b$$

$$= \frac{1}{56} \begin{bmatrix} 10 & -8 \\ -8 & 12 \end{bmatrix} \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -5 \\ 8 \\ 2 \end{bmatrix}$$

$$= \frac{1}{14} \begin{bmatrix} -57 \\ 47 \end{bmatrix}$$